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INVERSE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION FOR A TWO-DIMENSIONAL DOMAIN

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An approximate solution of a two-dimensional inverse problem is constructed on the basis of a solution of the Cauchy problem obtained in the form of a series in the derivatives and the Tikhonov regularization method.

The thermal state of power equipment is determined to a considerable extent by the heat-transfer characteristics on the surface of the structure elements. These conditions can often be found only from the solution of the inverse boundary-value problems of heat conduction. Such one-dimensional problems have been studied sufficiently completely [1]. However, the one-dimensional model cannot yield confident results for nonuniform heat delivery and thickness of the structure element.

Let us examine the problem of determining the temperature and heat fluxes from a heat-delivering boundary $y = W(x)$, $(0 < W(x) < d)$ of the two-dimensional domain $D = \{(x, y): x \in [0, d], y \in [0, W(x)]\}$ by means of known temperature measures and the law of heat transfer to the opposite side, which is given by the line $y = 0$. We consider the thermophysical parameters constant.

Let the curve $y = W(x)$ have a continuous external normal $n(x)$ and at points defined by the mesh $\omega_x = \{x_0 < x_1 < \ldots < x_k\}$ on the boundary $y = 0$ let the temperature $t(x, y, \tau)$ be known at the times $\omega_\tau = \{\tau_0 < \tau_1 < \ldots < \tau_p\}$, i.e.,

$$t(x_i, 0, \tau_j) = f_{ij}. \quad (1)$$

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The law of heat transfer from the environment at the surface y = 0 has the form

$$-\lambda \frac{\partial t(x,\ 0,\ \tau)}{\partial y} = \varphi(x,\ \tau;\ t(x,\ 0,\ \tau)).$$

Then by knowing (1) we can write

$$-\lambda \frac{\partial t(x_i,\ 0,\ \tau_i)}{\partial y} = q_{ij} \equiv \varphi(x_i,\ \tau_i;\ f_{ij}). \tag{2}$$

It is required to determine the change in temperature t_W and heat flux q_W on the surface y = W(x).

A Cauchy problem in the space variable

$$\frac{\partial t}{\partial \mathrm{Fo}} = \frac{\partial^2 t}{\partial \bar{x}^2} + \frac{\partial^2 t}{\partial \bar{y}^2},\ \left(\mathrm{Fo} = \frac{a\tau}{d^2},\ \bar{x} = \frac{x}{d},\ \bar{y} = \frac{y}{d}\right), \tag{3}$$

$$t(\bar{x},\ 0,\ \mathrm{Fo}) = f(\bar{x},\ \mathrm{Fo}), \tag{4}$$

$$-\frac{\lambda}{d}\ \frac{\partial t(\bar{x},\ 0,\ \mathrm{Fo})}{\partial \bar{y}} = q(\bar{x},\ \mathrm{Fo}) \tag{5}$$

corresponds to the formulated problem in the theory of heat conductivity. Here t, f, q are analytic, real functions. The existence and uniqueness of the solution of problem (3)-(5) in a certain neighborhood of the line $\bar{y}$ = 0 is given by the Kovalevskaya theorem. The solution is obtained easily in the form of a Maclauren series in the variable $\bar{y}$ [2]:

$$t = \sum_{n=0}^{\infty} \frac{\bar{y}^{2n}}{(2n)!} R^n f - \frac{\bar{y} d}{\lambda} \sum_{n=0}^{\infty} \frac{\bar{y}^{2n}}{(2n+1)!} R^n q, \tag{6}$$

where

$$R^n = \left(\frac{\partial}{\partial \mathrm{Fo}} - \frac{\partial}{\partial \bar{x}^2}\right)^n.$$

If the positive numbers r_1 and r_2 define the domain $D_1 = \{(\bar{x},\ \mathrm{Fo}): |\bar{x}| < r_1,\ |\mathrm{Fo}| < r_2\}$ of the analytic functions f and q, then problem (3)-(5) has a unique analytic solution (6) in the domain

$$\left\{(\bar{x},\ \bar{y},\ \mathrm{Fo}): (\bar{x},\ \mathrm{Fo}) \in D_1,\ |\bar{y}| < r_1 \sqrt{\frac{r_2}{r_2 + r_1^2}}\right\}. \tag{7}$$

Since function t is analytic in domain (7), then the series t'_x and t'_y converge in the same domain. However, the rates of convergence of the series t, t'_x, t'_y are distinct and degrade in the order listed.

Let the functions f and q belong to the class G of entire functions of exponential type of order I (entire trigonometric, exponential, hyperbolic, power functions, their combinations, polynomials, etc.). Then there are constants l, m such that

$$\left|\frac{\partial^{n+k} f}{\partial \bar{x}^n \partial \mathrm{Fo}^k}\right| < l^n m^k |f|,\ \left|\frac{\partial^{n+k} q}{\partial \bar{x}^n \partial \mathrm{Fo}^k}\right| < l^n m^k |q|\ \ (n,\ k = 0,\ 1,\ \ldots) \tag{8}$$

everywhere in the domain of definition of f and q. The series under consideration converge in this case for all $\bar{y}$, as follows from (7) for r_1, $r_2 \to \infty$. The subscript N will denote the N-th partial sum of the corresponding series. By using (8) it is easy to obtain an error estimate for the reduction of the series being studied:

$$\begin{gathered} E = \frac{|t_N - t|}{|t|} \sim \frac{\bar{y}^{2N+2}}{(2N+2)!} (m + l^2)^{N+1}, \\ E_x = \frac{|t'_{xN} - t'_x|}{|t'_x|} \sim \frac{\bar{y}^{2N+2}}{(2N+2)!} l (m + l^2)^{N+1}, \\ E_y = \frac{|t'_{yN} - t'_y|}{|t'_y|} \sim \frac{\bar{y}^{2N}}{(2N)!} (m + l^2)^N. \end{gathered} \tag{9}$$

Let us use the solution found for the Cauchy problem (3)-(5) to obtain approximate solutions of the inverse problem of heat conduction being considered by formulating conditions (1) and (2) of the inverse problem in conformity with conditions (4) and (5) of the Cauchy problem. Because of the inertia of the heat-transport process at points of the body remote from the heat delivering surface, the temperatures and heat fluxes are represented by smooth functions. In the case under consideration, points of the line $\bar{y} = 0$ are again most remote from the surface $\bar{y} = W/d$. Hence, the constraints ($f, q \in G$) imposed on the temperature f and heat flux q functions can be considered acceptable, and estimates (9) are valid.

Then the solution of Cauchy problem (3)-(5) permits an approximate determination of the change t_W and q_W on the opposite boundary $\bar{y} = W/d$ by means of the known f and q on the line $\bar{y} = 0$:

$$t_W = t_N \big|_{\bar{y}=\frac{1}{d}W(\bar{x})},$$

$$q_W = -\frac{\lambda}{d}[t'_{xN}\cos(\widehat{n,x}) + t'_{yN}\cos(\widehat{n,y})]_{\bar{y}=\frac{1}{d}W(\bar{x})}. \qquad (10)$$

The order of the relative error in the values of the functions t_W and q_W given by (10) agrees with E and E_y for $\bar{y} = W/d$, respectively.

As we see, to obtain the solution of the inverse problem it is required to differentiate functions known only from conditions (1) and (2) and with certain errors besides. It is well known that the numerical differentiation operator is unstable. To obtain the solution of the inverse boundary-value problem under investigation that is stable to perturbations in the functions f and q, it is necessary to use stable differentiation operators. Such operators can be constructed by different methods. Let us use the Tikhonov regularization method [3]. It is easy to see that the derivative $Z = \partial^{i+j}f/\partial x^i \partial\tau^j$ will be a solution of the integral equation

$$AZ = U, \qquad (11)$$

where

$$AZ = \begin{cases} \int\limits_{\tau_0}^{\tau} \frac{(\tau-\eta)^{j-1}}{(j-1)!} Z(x,\eta)\,d\eta, & i = 0,\ j \neq 0; \\ \int\limits_{x_0}^{x} \frac{(x-\xi)^{i-1}}{(i-1)!} Z(\xi,\tau)\,d\xi, & i \neq 0,\ j = 0; \\ \int\limits_{x_0}^{x}\int\limits_{\tau_0}^{\tau} \frac{(x-\xi)^{i-1}}{(i-1)!}\frac{(\tau-\eta)^{j-1}}{(j-1)!} Z(\xi,\eta)\,d\eta d\xi, & i \neq 0,\ j \neq 0; \end{cases}$$

$$U = \begin{cases} f - \sum\limits_{n=0}^{j-1} \frac{(\tau-\tau_0)^n}{n!}\frac{\partial^n f}{\partial\tau^n}\Big|_{\tau=\tau_0}, & i = 0,\ j \neq 0; \\ f - \sum\limits_{r=0}^{i-1} \frac{(x-x_0)^r}{r!}\frac{\partial^r f}{\partial x^r}\Big|_{x=x_0}, & i \neq 0,\ j = 0; \\ f - \sum\limits_{r=0}^{i-1} \frac{(x-x_0)^r}{r!}\frac{\partial^r f}{\partial x^r}\Big|_{x=x_0} - \sum\limits_{n=0}^{j-1}\frac{(\tau-\tau_0)^n}{n!} \times \\ \times \frac{\partial^n f}{\partial\tau^n}\Big|_{\tau=\tau_0} + \sum\limits_{r=0}^{i-1}\sum\limits_{n=0}^{j-1}\frac{(x-x_0)^r}{r!}\frac{(\tau-\tau_0)^n}{n!} \times \\ \qquad \times \frac{\partial^{r+n} f}{\partial x^r \partial\tau^n}\Big|_{\substack{x=x_0\\ \tau=\tau_0}}, \quad i \neq 0,\ j \neq 0. \end{cases}$$

Therefore, the problem of differentiating a function of two variables f reduces to solving a Volterra-type integral equation (11) of the first kind. In conformity with the Tikhonov regularization method,

$$Z^{\alpha}: M^{\alpha}[Z] = \|AZ - U\|^2_{L_2} + \alpha \|Z\|^2_{W^1_2} \to \min_Z \tag{12}$$

will be the stable algorithm of the solution of this incorrectly posed problem, where the regularization parameter $\alpha>0$ is selected in conformity with the error δ of giving the right sides of Eqs. (11).

The necessary condition for the minimum of the functional M^{α} results in the integro-differential equation

$$BZ + \alpha\left(Z - \frac{\partial^2 Z}{\partial x^2} - \frac{\partial^2 Z}{\partial \tau^2}\right) = C, \tag{13}$$

where

$$BZ = \begin{cases} \sum_{n=2}^{3} (-1)^n \int_{\tau_0}^{b_n} K_j(\tau, \varkappa, b_n) Z(x, \varkappa)\, d\varkappa, & i = 0; \\ \sum_{n=1}^{2} (-1)^{n+1} \int_{x_0}^{a_n} K_i(x, \psi, a_n) Z(\psi, \tau)\, d\psi, & j = 0; \\ \sum_{n=1}^{4} (-1)^{n+1} \int_{x_0}^{a_n} \int_{\tau_0}^{b_n} K_i(x, \psi, a_n) K_j(\tau, \varkappa, b_n) Z(\psi, \varkappa)\, d\varkappa\, d\psi; & \end{cases}$$

$$K_m(\beta, \gamma, \delta) = \frac{1}{(m-1)!} \sum_{r=0}^{m-1} C^r_{m-1} (\gamma - \beta)^{m-1-r} \frac{(\delta - \gamma)^{m+r}}{m+r};$$

$$C = \begin{cases} \int_{\tau}^{\tau_p} (\eta - \tau)^{j-1} U(x, \eta)\, d\eta, & i = 0; \\ \int_{x}^{x_k} (\xi - x)^{i-1} U(\xi, \tau)\, d\xi, & j = 0; \\ \int_{x}^{x_k} \int_{\tau}^{\tau_p} (\xi - x)^{i-1} (\eta - \tau)^{j-1} U(\xi, \eta)\, d\eta\, d\xi; & \end{cases}$$

a_n, b_n are components of the vectors (x_k, x, x, x_k), $(\tau_p, \tau_p, \tau, \tau)$, respectively.

Derivatives of the function f are recovered in the rectangular domain $\bar{D}_1 = \{(x, \tau): x \in [x_0, x_k], \tau \in [\tau_0, \tau_p]\}$. Equation (13) is defined in D_1. We consider the derivatives of Z to be given on the boundary of the domain D_1:

$$Z|_{\tau=\tau_0} = Z_{\tau_0}(x), \quad Z|_{\tau=\tau_p} = Z_{\tau_p}(x),$$
$$Z|_{x=x_0} = Z_{x_0}(\tau), \quad Z|_{x=x_k} = Z_{x_k}(\tau). \tag{14}$$

The derivatives are determined sequentially in order of increasing growth. This permits determination of more or less confident boundary conditions (14) for each specific case of recovering the derivative. Values of Z on the boundary of the domain D_1 are evaluated by finite differences by means of values found for the preceding derivatives.

The variational problem (12) therefore reduces to solving (13) in combination with the boundary conditions (14). This latter is accomplished by reduction to a system of $(k - 1)\cdot(p - 1)$ linear algebraic equations in the mesh $\omega = \omega_x \times \omega_\tau$. The system of equations obtained here for small values of the parameter α is poorly specified. Hence, it is preferable to solve it by one of the iteration methods. The regularization parameter α was selected by the residual principle [4].

Therefore, by realizing the variational problem (12) on recovery of the derivatives of the function f sequentially (starting with the lowest derivatives), and using the values of the lowest derivatives already obtained for the computation of the right sides of (11) and the boundary conditions (14), we have obtained a stable algorithm for differentiation of an experimental function of two variables.

To assure the required accuracy in the differentiation, it is necessary to use a compact mesh ω. However, the order of the system of linear algebraic equations will hence be

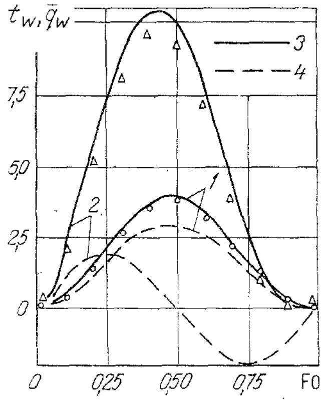

Fig. 1

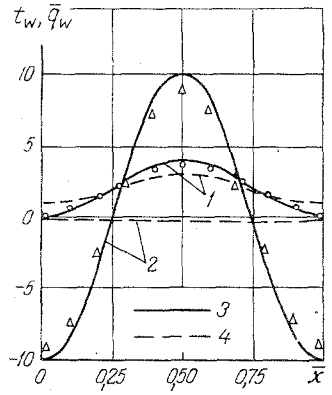

Fig. 2

Fig. 1. Results of solving a model inverse problem in the section $\bar{x} = 0.5$: 1) temperature t_W; 2) heat flux $\bar{q}_W$; 3) exact values of the quantities; 4) results for a one-dimensional model of the inverse problem; results of solution of the two-dimensional problem are displayed by dots ($\delta = 0.05$).

Fig. 2. Results of the solution of the model inverse problem for Fo = 0.5. Notation the same as in Fig. 1.

large. The specificity of the system is degraded noticeably with the increase in order. The order of the system to be solved can be lowered considerably by considering the differentiation problem not in the whole domain D_1 but in a certain finite covering $\{D_1^{ij}\}$ ($i = 1, \ldots, I$; $j = 1, \ldots, J$). This latter is selected from the following conditions: 1) the union of all D_1^{ij} agrees exactly with D_1; 2) the neighboring domains D_1^{ij} have three common layers of nodes of the mesh ω at their contact; 3) the order of the system of algebraic equations in each domain D_1^{ij} does not exceed a certain optimal value M. The second condition permits elimination of the error in giving the boundary conditions (14) for the boundary points of the domains D_1^{ij} that are internal points of the domain D_1. Realization of the variational problem (12) is performed sequentially for each domain D_1^{ij}. For each succeeding domain D_1^{ij} from the domains where the computations have already been performed, the values obtained for the function Z are taken as boundary conditions. The optimal order of the system of algebraic equations is selected with the characteristics of the electronic computer to be used taken into account (operational memory, fast-response). Thus the optimal order for the BESM-4M is $M \sim 40$. In the approach being proposed, the variational problem (12) should be solved $I \times J$ times for each derivative to be recovered. However, the possibility hence appears of considering a sufficiently large array of experimental information, and the specificity of the systems being considered is also improved substantially.

Therefore, the stable algorithm described for numerical differentiation of the function of two variables in combination with (10) determines an approximate solution, stable to perturbations in the initial function f, for the two-dimensional inverse boundary value problem of heat conduction, whose confidence is determined by the estimates (9). Values of the constants l and m required for this are found from (8) by means of the recovered derivatives of the functions f and q. Since information about the functions f and q is constrained only by the conditions (1) and (2), then the recovery of higher order derivatives of these functions is difficult because of the cumulative error in the right sides of (11). Hence, it is expedient to take N = 2 to reduce the series in the solution of the Cauchy problem (6).

The above constrains the applicability of the proposed solution of the inverse problem for rapidly proceeding thermal processes. Thus, the error in the results of this approach reached 50% and more in the initial period under thermal impact. However, as methodological computations showed, the algorithm described is quite effective for a moderate change in the temperature.

The algorithm for the solution of the two-dimensional inverse problem of heat conduction under consideration is realized in ALGOL. Results of the solution for the domain $D_1 = \{0 \leqslant \bar{x} \leqslant 1, 0 \leqslant \bar{y} \leqslant 0{,}2\}$ of the following model problem are shown in Figs. 1 and 2. The surfaces $\bar{y} = 0$, $\bar{x} = 0$ and $\bar{x} = 1$ are heat insulated and the heat flux $\bar{q}_W$ is delivered to the surface

$\bar{y} = 0.2$ and assures the following temperature change on the surface $\bar{y} = 0$:

$$t|_{\bar{y}=0} = f(\bar{x}, \text{Fo}) = [1 - 0{,}5\cos(2\pi\bar{x})]\sin(\pi\,\text{Fo}).$$

The exact solution of the direct problem under consideration is obtained easily from the solution of the Cauchy problem (7) for the mentioned kind of function f and q = 0. The temperature change

$$t(\bar{x}_i, 0, \text{Fo}_j) = f_{ij} \equiv (1 + \delta\varepsilon_{11\times i+j}) \times f(\bar{x}_i, \text{Fo}_j), \ (i, j = 0, \ldots, 10),$$

is given on the heat insulated surface $\bar{y} = 0$ for the model inverse problem, where $\bar{x}_i = \text{Fo}_i = 0.1 \times i$; $\{\varepsilon_{11\times i+j}\}$ is a sequence of random numbers distributed uniformly in the interval $[-1, 1]$, and δ is the maximum relative error with which the temperature is known on the surface $\bar{y} = 0$.

As we see from the graphs, utilization of a one-dimensional inverse problem model does not permit recovery of the heat flux in this example. The errors in solving the two-dimensional inverse problem for different realizations of the random numbers ε did not exceed 5% for the temperature and 10% for the heat flux for $\delta = 0.05$. Methodological computations performed displayed the stability of the algorithm described for the solution of the two-dimensional inverse boundary-value problem of heat conduction quite well for perturbations of the initial data (1) and (2).

NOTATION

d, characteristic dimension of the plane domain under investigation; n, outer normal to the boundary $\bar{y} = (1/d)W(\bar{x})$; τ, time; Fo, Fourier criterion; $\bar{x}$, $\bar{y}$, dimensionless coordinates; t, dimensionless temperature; $t'_{\bar{x}}$, $t'_{\bar{y}}$, derivatives of the function t with respect to $\bar{x}$ and $\bar{y}$, respectively; λ, heat-conduction coefficient; a, thermal diffusivity; C^r_{m-1}, binomial coefficients.

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